Journal of Statistical Physics, Vol. 53, Nos. 5/6, 1988

Derivation of a Hydrodynamic Equation for Ginzburg–Landau Models in an External Field

József Fritz[1,2] **and Christian Maes**[1,3]

Received April 19, 1988

The lattice approximation to a time-dependent Ginzburg–Landau equation is investigated in the presence of a small external field. The evolution law conserves the spin, but is not reversible. A nonlinear diffusion equation of divergence type is obtained in the hydrodynamic limit. The proof extends to certain stochastically perturbed Hamiltonian systems.

KEY WORDS: Ginzburg–Landau models; hydrodynamic limit; perturbation of parabolic equations; interpolation and singular integrals.

1. INTRODUCTION

In the last 10 years the problem of giving a microscopic foundation to the hydrodynamic behavior of interacting particle systems has received a great deal of interest. There is a fairly well developed theory in the case of lattice gases and related models; see the survey paper by De Masi *et al.*[1] and Lebowitz *et al.*[2] for more recent results. The situation is different for systems with continuous trajectories. The most fundamental examples of this kind are the Hamiltonian models of classical mechanics. While more or less explicit calculations are available in the case of one-dimensional hard rods and harmonic oscillators,[3,4] there is no method to treat less degenerate situations. Stochastic models, on the other hand, allow for a more complete discussion. It is therefore interesting to study them although they describe only some particular features of physical reality.

The main purpose of this paper is to extend the results of ref. 5 to

[1] Department of Mathematics and Physics, Rutgers University, New Brunswick, New Jersey 08903.

[2] Permanent address: Mathematical Institute, HAS, H-1364 Budapest, Pf. 127, Hungary.

[3] Aspirant NFWO, Instituut voor Theoretische Fysika, K.U. Leuven, Belgium.

0022-4715/88/1200-1179$06.00/0

Ginzburg–Landau models with external field. Such phenomenological (quasimicroscopic) models are widely used in low-temperature physics; see refs. 6 and 7 for further references to the physics literature. Driven diffusive systems have been treated in the context of lattice gases.[8-11] The methods developed here are applicable also to stochastically perturbed anharmonic systems. We present the methods in the simplest nontrivial cases; some possible extensions are discussed at the end of Sections 1–2 and 3.

We start by describing the physical context of some simple microscopic dynamics [Eqs. (1.4) and (1.16) below]. See also ref. 12.

We consider a one-dimensional lattice system of continuous spins. We think of the sites $x \in Z$ as being the center of a unit cell in R. To each site $x \in Z$ we associate a real-valued spin $S(x)$ which describes the density in the cell around x. Each cell is in thermal equilibrium but still small enough for the free energy $V(S(x))$ to vary in space:

$$H(S) = \sum_{x \in Z} V(S(x)) \tag{1.1}$$

V is some real-valued convex function. We can define a chemical potential

$$m_x = \frac{\delta H(S)}{\delta S(x)} = V'(S(x)) \tag{1.2}$$

in every cell $x \in Z$. As usual a difference in potential between neighboring cells $x+1$ and x gives rise to a current from x to $x+1$ over a time interval dt

$$dj_t(x) = \tfrac{1}{2}(m_x - m_{x+1})\, dt \tag{1.3}$$

To account for the inaccuracy of this (reduced) description, we introduce a random uncorrelated current $dW(t, x)$ associated to each bond $\{x, x+1\}$. $W(t, x)$ is a Wiener process and represents the random current flowing from site x to $x+1$. Finally, we wish to add an *a priori* prescribed current $\varepsilon J(S(x))$. It represents the net current from x to $x+1$ due to an external field. εJ is a real-valued function and should be considered as the product of a constant electric field and the conductivity, which depends on the local configuration. The coefficient $0 < \varepsilon < 1$ measures the strength of the electric field and will be thought of as being very small.

The dynamics is governed by the equation expressing conservation of mean density in the presence of a source:

$$dS_t(x) + \operatorname{div} d\gamma_t(x) = -\varepsilon \operatorname{div} J(S_t(x))\, dt \tag{1.4}$$

with $\operatorname{div} f(x) = f(x) - f(x-1)$ for a scalar function f on Z, and $d\gamma_t(x) = dj_t(x) + dW(t, x)$. The initial condition is specified by $S_0(x) = \sigma(x)$, a given configuration of spins on the lattice Z.

We ask now how to obtain from these (stochastic) microscopic equations a macroscopic description of the process. It is clear that the main ingredient will be a rescaling of extensive quantities for which the hydrodynamics is investigated. The procedure is called the hydrodynamic scaling limit. We have to rescale space and time using the small parameter ε:

$$t \to t/\varepsilon^2, \qquad x \to x/\varepsilon \tag{1.5}$$

The rescaled density field $S_t^\varepsilon(x)$ is defined on the lattice εZ by

$$S_t^\varepsilon(x) = S_{t/\varepsilon^2}(x/\varepsilon) \tag{1.6}$$

Equation (1.4) is now changed into

$$dS_t^\varepsilon(x) = \tfrac{1}{2}\Delta_\varepsilon V'(S_t^\varepsilon(x))\,dt + \nabla_\varepsilon^* J(S_t^\varepsilon(x))\,dt + \nabla_\varepsilon^*\,dW_\varepsilon(t,x) \tag{1.7}$$

with

$$S_0^\varepsilon(x) = \sigma^\varepsilon(x) = \sigma(x/\varepsilon)$$

as initial condition. $\Delta_\varepsilon = -\nabla_\varepsilon^*\nabla_\varepsilon$ is the usual lattice Laplacian, i.e.,

$$\Delta_\varepsilon f(x) = \varepsilon^{-2}(f(x+\varepsilon) - 2f(x) + f(x-\varepsilon))$$

$$\nabla_\varepsilon f(x) = \varepsilon^{-1}(f(x+\varepsilon) - f(x))$$

$$\nabla_\varepsilon^* f(x) = \varepsilon^{-1}(f(x-\varepsilon) - f(x))$$

for some scalar function $f(x)$, $\quad x \in R$

$$W_\varepsilon(t,x) = \varepsilon W(t/\varepsilon^2, x/\varepsilon) \tag{1.8}$$

The stochastic differential equation (1.7) is the starting point of our discussion. If $J=0$, then this process has a family of local equilibrium distributions. They are the Gibbs states $\mu_{\lambda,\varepsilon}$ with energy function

$$H_{\lambda,\varepsilon}(S) = \sum_{x\in\varepsilon Z} V(S(x)) - \lambda^\varepsilon(x)\,S(x) \tag{1.9}$$

where $\lambda(\cdot)$ is a smooth profile. (The superscript ε indicates that the function has been made into a step function of size ε on R; this will be explained more precisely in the next section.) It was shown in ref. 5 that the time-evolved expectations can be calculated in an asymptotic sense by means of time-dependent measures $\mu_{\lambda,\varepsilon}$. The stationary measures for the case where $J \neq 0$ are not known, but the leading term $\frac{1}{2}\Delta_\varepsilon V'$ dominates the external field because of their different scaling.[8,9,13] We will show that, despite the fact that the $\mu_{\lambda,\varepsilon}$ are no longer a family of local equilibrium measures, macroscopic expectations are still calculable using the $\mu_{\lambda,\varepsilon}$ with a time-dependent profile $\lambda = \lambda(t, x)$.

Assume that the initial configuration σ^ε approaches some smooth function ρ_0. The hydrodynamic equation that will be derived from (1.7) for the density field $S_t^\varepsilon(x)$ in the limit $\varepsilon \downarrow 0$ has the form

$$\begin{aligned} \partial_t \rho(t, x) &= \partial_x[\tfrac{1}{2} D(\rho(t, x))\, \partial_x \rho(t, x) + B(\rho(t, x))] \\ \rho(0, \cdot) &= \rho_0(\cdot) \end{aligned} \tag{1.10}$$

It is a nonlinear diffusion equation. The assumptions on V and J will guarantee that D is a strictly positive and bounded function on R, and that B has a bounded derivative. Both D and B will be defined in (2.21). If V is a quadratic potential, then $D(\cdot)$ is a (diffusion) constant. If $J = 0$, then $B(\cdot) = 0$ and the equation reduces to the one obtained in ref. 5. If J is linear, then $B(\rho) \sim \rho$.

Our treatment of system (1.7) extends also to vector models on Z^d. We briefly indicate how to generalize (1.4). At every site $x \in Z^d$ there is a vector-valued spin, $S(x) \in R^n$. The infinite configuration is $S = \{S(x), x \in Z^d\}$, with corresponding free energy $H(S)$, For example,

$$H(S) = \sum V(S(x)) \tag{1.11}$$

where $V\colon R^n \to R$ is strictly convex. To each bond $\{x, y\}$ of nearest neighbors $x, y \in Z^d$ we associate a vector current $J_{xy} = -J_{yx} \in R^n$ which is a local function of the configuration S, and a standard Wiener (vector)process $W_{xy} = -W_{yx} \in R^n$. Again, the evolution is given by the continuity equation [as in (1.4)]

$$dS_x + \sum_{|y-x|=1} [d\gamma_t(x \to y) + \varepsilon J(S)\, dt] = 0 \tag{1.12}$$

where

$$d\gamma_t(x \to y) = C_{xy}\, dW_{xy} + \frac{1}{2} C_{xy} C_{xy}^* \left(\frac{\delta H}{\delta S(x)} - \frac{\delta H}{\delta S(y)}\right) \tag{1.13}$$

C_{xy} is a symmetric $n \times n$ matrix.

This generalization allows us to incorporate some stochastic perturbations of Hamiltonian systems. Consider, for example, the case $d = 1$, $n = 2$. In that case we have two real coordinates, the momentum $p(x)$ and the position $r(x)$, at each site $x \in Z$. We repeat the same discussion as above for the vector $S(x) = (p(x), r(x))$. Instead of (1.1) we take

$$H(S) = \sum p^2(x)/2 + V(r(x)) \tag{1.14}$$

where V is a real-valued symmetric convex function. For the (vector) current we choose

$$J(S)(x)) = (-V'(r(x)),\ -p(x+1)) \tag{1.15}$$

The continuity equation (1.4) can be written down explicitly. Let $p_t^\varepsilon(x)$ and $r_t^\varepsilon(x)$ denote the rescaled momentum and density, respectively [see (1.6)]. The evolution is given by a coupled system of stochastic equations:

$$\begin{aligned} dp_t^\varepsilon(x) &= -\nabla_\varepsilon^* V'(r_t^\varepsilon(x))\,dt + \frac{\alpha}{2}\Delta_\varepsilon p_t^\varepsilon(x)\,dt + \sqrt{\alpha}\,\nabla_\varepsilon^*\,dW_\varepsilon(t,x) \\ dr_t^\varepsilon(x) &= \nabla_\varepsilon p_t^\varepsilon(x)\,dt + \frac{\beta}{2}\Delta_\varepsilon V'(r_t^\varepsilon(x))\,dt + \sqrt{\beta}\,\nabla_\varepsilon^*\,d\tilde{W}_\varepsilon(t,x) \end{aligned} \tag{1.16}$$

with initial conditions p_0^ε and r_0^ε. We used

$$C_{xy} = \begin{pmatrix} \sqrt{\alpha} & 0 \\ 0 & \sqrt{\beta} \end{pmatrix}$$

α and β are arbitrary positive numbers. $W_\varepsilon(t,x)$ and $\tilde{W}_\varepsilon(t,x)$ are independent Wiener processes for each $x \in \varepsilon Z$.

This model represents a stochastically perturbed anharmonic system. Indeed, consider the Hamiltonian

$$E(p,q) = \sum p^2(x)/2 + V(q(x) - q(x-1)) \tag{1.17}$$

where p and q are the canonical coordinates. Introducing $r(x) = q(x+1) - q(x)$, we obtain the Hamilton equations

$$\begin{aligned} dp_t(x) &= -[V'(r_t(x-1)) - V'(r_t(x))]\,dt \\ dr_t(x) &= [p_t(x+1) - p_t(x)]\,dt \end{aligned} \tag{1.18}$$

After a (hyperbolic!) rescaling of these equations, $t \to t/\varepsilon$ and $x \to x/\varepsilon$, we obtain (1.16) with $\alpha = \beta = 0$, where now $p_t^\varepsilon(x) = p_{t/\varepsilon}(x/\varepsilon)$ and $r_t^\varepsilon(x) = r_{t/\varepsilon}(x/\varepsilon)$. The structure of the additional terms in (1.16) (damping and noise) is determined by the requirement of reversibility. In particular, the Gibbs states with energy $E(p,r) = \sum p^2(x)/2 + V(r(x))$ are stationary. Moreover, it is easy to check that p and r are conserved quantities of (1.18). The stochastic term is responsible for the breaking of conservation of energy.

In this way we established a connection between the stochastically perturbed anharmonic system and a vector-valued Ginzburg–Landau model in an external field.

The local equilibrium states of this problem are given by the energy function,

$$H_{\gamma,\lambda}(p, r) = \sum p^2(x)/2 + V(r(x)) - \gamma^\varepsilon(x)\, p(x) - \lambda^\varepsilon(x)\, r(x) \tag{1.19}$$

with $\gamma^\varepsilon(x)$ and $\lambda^\varepsilon(x)$ defined as in (2.11). Of course, the stochastic perturbation destroys the Hamiltonian structure. Nevertheless, (1.16) still yields some physical information. After taking the hydrodynamic limit we obtain the (macroscopic) equations for the momentum field Π and the density field ρ, which are of the following form:

$$\begin{aligned} \partial_t \Pi(t, x) &= D(\rho)\, \partial_x \rho(t, x) + \frac{\alpha}{2} \partial_{xx}^2 \Pi(t, x) \\ \partial_t \rho(t, x) &= \partial_x \Pi(t, x) + \frac{\beta}{2} \partial_x (D(\rho)\, \partial_x \rho(t, x)) \end{aligned} \tag{1.20}$$

with initial conditions $\Pi(0, x) = \Pi_0(x)$ and $\rho(0, x) = \rho_0(x)$. $D(\rho)$ is the same as in (1.10).

Observe that Eqs. (1.20) reduce to a nonlinear wave equation,

$$\partial_{tt}^2 \rho(t, x) = \partial_x (D(\rho)\, \partial_x \rho(t, x)) \tag{1.21}$$

by formally letting α, β go to zero.

In the next two sections we repeat in a more precise way the main points of this introduction. The other sections are devoted to the proof of the results in the simplest case (as far as the notation is considered); a more complete description of the contents of this paper is presented at the end of Section 3.

2. GINZBURG–LANDAU MODELS WITH EXTERNAL FIELD. MAIN RESULT

We wish to formulate the problem of deriving the hydrodynamic equations in terms of a family of Markov processes, $\{S_t^\varepsilon,\ t \geqslant 0,\ \varepsilon > 0\}$. It is convenient to embed the configuration space for each $\varepsilon > 0$ into a space of functions on R.

We *define*, therefore,

$$L_e^2 = \bigcap_{r>0} L^2(R, \theta_r(x)\, dx) \tag{2.1}$$

as the real Hilbert space of locally integrable functions $f\colon R \to R$ with scalar product

$$\langle f, g \rangle_r = \int \theta_r(x)\, f(x)\, g(x)\, dx \tag{2.2}$$

and norm

$$|f|_r^2 = \int \theta_r(x) f^2(x)\, dx \tag{2.3}$$

The weight function θ_r is defined in detail in ref. 5, but we can think of it as $\theta_r(x) = \exp(-r\,|x|)$. If $|\sigma_n - \sigma|_r \to 0$, $\forall r > 0$, then $\sigma_n \to \sigma$ in the strong topology of L_e^2. The weak topology of L_e^2 is given by a fundamental system of the neighborhoods of $0 \in L_e^2$, namely,

$$U_\alpha(\phi_1,..., \phi_k) = \{\sigma \in L_e^2;\ |\phi_i(\sigma)| < \alpha,\ \forall i\colon 1,..., k\} \tag{2.4}$$

with $\alpha > 0$, $k \in N_0$, and $\phi_i \in L_e^{2*}$. Note that

$$L_e^{2*} = \bigcup_{r>0} L^2(R, \theta_{-r}(x)\, dx) \tag{2.5}$$

and, if $\sigma \in L_e^2$, $\phi \in L_e^{2*}$, then

$$\phi(\sigma) = \int \phi(x)\, \sigma(x)\, dx \tag{2.6}$$

L_e^2 is a reflexive space and its balls,

$$B(\{b_r\}) = \{\sigma \in L_e^2;\ |\sigma|_r \leqslant b_r \text{ for all } r > 0\} \tag{2.7}$$

where $\{b_r\}_{r>0}$ are positive numbers, are weakly compact. For $X \subset L_e^2$, we define

$$C_s(X) \qquad \text{and} \qquad C_w(X) \tag{2.8}$$

respectively as the spaces of strongly and weakly continuous and bounded maps of X into R. If X is a convex set, we define $D_s(X)$ as the space of functional differentiable functions f on X, i.e. if σ, $\bar{\sigma} \in X$, and $\delta = \sigma - \bar{\sigma}$, then

$$f(\sigma) - f(\bar{\sigma}) = \int_0^1 \int \delta(x)\, Df(x, \bar{\sigma} + q\delta)\, dx\, dq \tag{2.9}$$

where $Df\colon X \to L_e^{2*}$ is strongly continuous. The space of weakly differentiable $\lambda \in L_e^2$ with derivative belonging to L_e^2 will be denoted by H_e^1.

The *embedding* of the configuration spaces into L_e^2 goes as follows. Define for $\varepsilon > 0$ the intervals

$$C_\varepsilon(x) = [\varepsilon(x - 1/2), \varepsilon(x + 1/2)), \qquad x \in Z \tag{2.10}$$

For $y \in R$, let y^ε denote the unique integer $x \in Z$ for which $y \in C_\varepsilon(x)$: $y^\varepsilon = x$. For example, $(\varepsilon x)^\varepsilon = x$ for all $x \in Z$. If both y_1, $y_2 \in C_\varepsilon(x)$ for the same $x \in Z$, then obviously

$$y_1^\varepsilon = y_2^\varepsilon = x \tag{2.11}$$

Define Ω_ε as the space of all functions $f \in L_e^2$, which are constant on the intervals $C_\varepsilon(x)$, $x \in Z$. Given a function $f \in L_e^2$, we define the step function $f^\varepsilon = I_\varepsilon f \in \Omega_\varepsilon$:

$$I_\varepsilon f(y) = \varepsilon^{-1} \int_{C_\varepsilon(y^\varepsilon)} f(z)\, dz \tag{2.12}$$

This implies that $\Omega_\varepsilon = I_\varepsilon L_e^2$. On the other hand, given a function σ on Z, we define the step function σ^ε on R by taking

$$\sigma^\varepsilon(y) = \sigma(y^\varepsilon) \tag{2.13}$$

It is clear that these definitions imply that, if $\sigma^\varepsilon \in \Omega_\varepsilon$, then

$$\phi(\sigma^\varepsilon) = \int \phi(x)\, \sigma^\varepsilon(x) = \varepsilon \sum_{x \in Z} \phi^\varepsilon(\varepsilon x)\, \sigma(x) \tag{2.14}$$

for a test function ϕ.

The microscopic dynamics is determined by (1.7). V: $R \to R$ is a convex function and J: $R \to R$ is a continuously differentiable function such that

$$0 < c_1 \leqslant V''(x) \leqslant c_2 \tag{2.15}$$

with c_1, c_2 absolute constants and V''', J', and J'' are bounded.

Since multiplying V by a constant does not change the problem, we may and do assume that $c_1 = 1 + \alpha$, $c_2 = 1 - \alpha$ with $\alpha \in (0, 1)$.

The conditions immediately imply (see ref. 5 for a more detailed discussion) that there is a solution $S_t = S_t(\cdot, \sigma)$ to (1.4) with initial condition $S_0 = \sigma \in \Omega = \Omega_1$. In fact, the SDE (1.4) gives rise to a Markov process on Ω. The process $S_t^\varepsilon \in \Omega_\varepsilon$ is defined by

$$S_t^\varepsilon(y) = S_{t/\varepsilon^2}(y^\varepsilon) \qquad \text{for all} \quad y \in R, \quad t \geqslant 0 \tag{2.16}$$

The initial value is $S_0^\varepsilon = \sigma^\varepsilon \in \Omega_\varepsilon$. The associated Markov semigroup P_ε^t on $C_s(\Omega_\varepsilon)$ is defined as

$$P_\varepsilon^t g(\sigma) = E_\sigma[g(S_t^\varepsilon)] = E[g(S_t^\varepsilon) \mid S_0^\varepsilon = \sigma] \qquad \text{for} \quad g \in C_s(\Omega_\varepsilon) \tag{2.17}$$

$E[\cdot \mid S_0^\varepsilon = \sigma] = E_\sigma[\cdot]$ is the expectation with respect to the process started from

$$S_0^\varepsilon = \sigma \in \Omega_\varepsilon \tag{2.18}$$

The limiting evolution equation was introduced in (1.10). Define for $\lambda \in R$

$$\begin{aligned} q_\lambda(dy) &= Z_\lambda^{-1} \exp[-V(y) + \lambda y]\, dy \\ Z_\lambda &= \int \exp[-V(y) + \lambda y]\, dy \\ F(\lambda) &= \log Z_\lambda \\ A(\lambda) &= -\int J(y)\, q_\lambda(dy) \end{aligned} \tag{2.19}$$

Consider the equations,

$$\begin{aligned} \rho(0, x) &= \rho_0(x) \\ \rho(t, x) &= F'(\lambda(t, x)) \\ F''(\lambda(t, x))\, \partial_t \lambda(t, x) &= \tfrac{1}{2} \partial_{xx}^2 \lambda(t, x) + \partial_x A(\lambda(t, x)) \end{aligned} \tag{2.20}$$

Equation (1.10) can be recovered from this by noticing that F' is strictly increasing (see Section 4, proof of Lemma 4). Hence, if $\rho = F'(\lambda)$, then

$$D(\rho) = 1/F''(\lambda) \qquad \text{and} \qquad B(\rho) = A(\lambda) \tag{2.21}$$

are well defined for all $\rho \in R$. The inverse function of F' will be denoted by Q:

$$\lambda = Q(\rho) \tag{2.22}$$

Notice that if G denotes the convex conjugate (Legendre transform) of F, i.e., $G(\rho) = \sup_\lambda [\rho\lambda - F(\lambda)]$, then $Q = G'$.

Equations (1.10) and (2.20) are uniquely solved in the weak sense (see Section 4, Lemma 4):

Theorem 1. If σ^ε converges weakly in L_e^2 to some $\rho_0 \in H_e^1$ as $\varepsilon \to 0$, then $\phi(S_t^\varepsilon)$ converges in probability to $\phi(\rho(t, \cdot))$ for each $t > 0$ and $\phi, \phi' \in L_e^{2*}$, where $\rho(t, \cdot)$ is the uniquely defined weak solution to the limiting equation (1.10).

Idea of the proof. We will first prove the theorem in the case of random initial data (Section 7). We assume that the initial configuration σ^ε on εZ is distributed by a product measure $\mu_{\lambda,\varepsilon}$:

$$\mu_{\lambda,\varepsilon}(d\sigma) = \prod_{x \in \varepsilon Z} q_{\lambda^\varepsilon(x)}(d\sigma(x)) \tag{2.23}$$

where $\lambda(\cdot) \in H_e^1$ is a given profile function for the chemical potential and $\lambda^\varepsilon = I_\varepsilon \lambda$.

Using the embedding (2.13) and the definition (2.19), this is clearly equivalent with saying that the initial configuration σ on Z is distributed by a product measure

$$\sim \prod_{x \in Z} \exp[-V(\sigma(x) + \lambda^{\varepsilon}(\varepsilon x)\, \sigma(x)]$$

In particular,

$$\int \sigma(x)\, \mu_{\lambda,\varepsilon}(d\sigma) = F'(\lambda^{\varepsilon}(x)) \tag{2.24}$$

where $F(\cdot)$ was defined in (2.19). Moreover, using the embedding (2.12) and (2.13), we can define $\mu_{\lambda,\varepsilon}$ on the Borel field of L^2_e, with $\mu_{\lambda,\varepsilon}(\Omega_\varepsilon) = 1$. The following remarks motivate the role of $\mu_{\lambda,\varepsilon}$ in the derivation of the limiting equation.

Let $\psi \in C_0^\infty(R^k)$, $\phi_i \in C_0^\infty(R)$ for i: 1,..., k, $k \in N_0$, and define

$$f(\sigma) = \psi(\phi_1(\sigma),..., \phi_k(\sigma)) \tag{2.25}$$

If G_ε is the generator of the semigroup P_ε^t [see Eq. (2.17)], then, after an integration by parts,[5,12]

$$\int G_\varepsilon f(\sigma)\, \mu_{\lambda,\varepsilon}(d\sigma)$$
$$= -\iint [\tfrac{1}{2}\nabla_\varepsilon \lambda^{\varepsilon}(x) - J(\sigma(x))]\, \nabla_\varepsilon\, Df(x, \sigma)\, dx\, \mu_{\lambda,\varepsilon}(d\sigma) \tag{2.26}$$

with functional derivative

$$Df(x, \sigma) = \sum_{i=1}^{k} \partial_i \psi(\phi_1(\sigma),..., \phi_k(\sigma))\, \phi_i(x) \tag{2.27}$$

Furthermore, by the law of large numbers (to be discussed in Section 6)

$$\lim_{\varepsilon \downarrow 0} \int f(\sigma)\, \mu_{\lambda,\varepsilon}(d\sigma) = f(\rho_0)$$

and

$$\lim_{\varepsilon \downarrow 0} \int G_\varepsilon f(\sigma)\, \mu_{\lambda,\varepsilon}(d\sigma)$$
$$= -\int [\tfrac{1}{2}\lambda'(x) + A(\lambda(x))]\, \nabla Df(x, F'(\lambda))\, dx$$
$$= \frac{d}{dt} f(\rho(t, \cdot))_{t=0} \tag{2.28}$$

if ρ solves the limiting equation (1.10) and $\rho_0 = F'(\lambda)$. This suggests that we can follow the resolvent method (Section 7) as outlined in ref. 12 to get an expression similar to (2.28) for positive times. In this approach we extend (2.28) to functions of the type

$$f_{z,\varepsilon}(\sigma) = \int_0^\infty e^{-zt} P_\varepsilon^t g_\phi(\sigma)\, dt, \qquad z > 0, \quad \sigma \in \Omega_\varepsilon$$
$$g_\phi(\sigma) = \phi(\sigma), \qquad \phi, \phi' \in L_e^{2*} \tag{2.29}$$

Since the balls of L_e^2 are weakly compact, the Stone-Weierstrass theorem allows us to extend the law of large numbers (2.28) from functions of type (2.25) to weakly continuous functions. Therefore we have to show that f and ∇Df are weakly continuous functions of the initial configuration σ. Since everything depends on the scaling parameter, this continuity should be uniform in ε. The dependence on the initial configuration can be expressed via the functional derivative as defined in (2.27). The variation of $f_{z,\varepsilon}$ is given by

$$Df_{z,\varepsilon}(x, \sigma) = \int_0^\infty e^{-zt} DP_\varepsilon^t g_\phi(x, \sigma)\, dt$$
$$= E_\sigma \left[\int_0^\infty e^{-zt} \int p_{a,b}(0, t; x, y)\, \phi(y)\, dy\, dt \right] \tag{2.30}$$

where $p_{a,b}$ is the fundamental solution to

$$\partial_t u_t = \nabla_\varepsilon^*(b(t, x)\, u_t) + \tfrac{1}{2} \Delta_\varepsilon(a(t, x)\, u_t)$$
$$a(t, x) = V''(S_t^\varepsilon(x)), \qquad b(t, x) = J'(S_t^\varepsilon(x)) \tag{2.31}$$

with initial condition

$$p_{a,b}(s, s; x, y) = 1/\varepsilon \qquad \text{if} \quad x^\varepsilon = y^\varepsilon$$
$$= 0 \qquad \text{else.} \tag{2.32}$$

(2.31) is the first variational system of (1.7). The crucial step is now of course taking the limit $\varepsilon \downarrow 0$ in (2.29): this is the problem of smooth dependence of solutions to the SDE (1.7) on initial data. To pass to the limiting equation, various compactness properties of these families of functions $(\{f_{z,\varepsilon}\}, \{\nabla_\varepsilon Df_{z,\varepsilon}\}, \{\int P_\varepsilon^t g_\phi(\sigma)\, \mu_{\lambda,\varepsilon}(\sigma)\})$ will have to be investigated. To establish the weakly continuous dependence of solutions to our dynamics on the initial configuration we can concentrate on the regularity properties of the fundamental solution $p_{a,b}$ as it appears in (2.30). In Section 5 we will show that the study of $p_{a,b}$ can be reduced to that of $p_{a,b=0}$ and the method of singular integrals (as explained in refs. 5 and 14) will be applied. The singularity caused by $\nabla_\varepsilon^*(bu_t)$ can be handled by means of an L_r^4 estimate (see Section 4, Lemma 3). In view of (2.30), the basic object of our studies is an integral operator $P_{a,b}$ defined for measurable h: $[0, \infty) \times R \to R$ and $z > 0$ [see also (5.14)]:

$$P_{a,b}h(s, x) = \int_s^\infty e^{z(s-t)} \int p_{a,b}(s, x; t, y)\, h(t, y)\, dy\, dt$$

In particular, to prove that $\nabla_\varepsilon\, Df$ is weakly continuous, we have to show that $\Delta_\varepsilon P_{a,b}$ is a uniformly (in ε) bounded map of $L^q([0, \infty) \times R)$ into itself for some $q > 2$. This is the most difficult step of the proof. The case $b = 0$ (i.e., for P_a) was solved in ref. 5. On the other hand,

$$P_{a,b}h(s, x) = P_a h(s, x) + P_a b(\cdot, \cdot)\nabla_\varepsilon P_{a,b}h(s, x) \tag{5.15}$$

where b is a multiplication operator, is a standard perturbative identity. In Section 4 we extend the energy inequality to an L^4 norm. Therefore, $\nabla_\varepsilon P_{a,b}$ maps $L^4([0, \infty) \times R)$ into itself, whence the desired bound of $\Delta_\varepsilon P_{a,b}$ follows by (5.15) and interpolation (see ref. 5, the Appendix of ref. 14, and refs. 15 and 16). These bounds are sufficient to derive that the resolvent $f_{z,\varepsilon}$ converges to that of the limiting equation, whence

$$\lim_{\varepsilon \downarrow 0} \int P_\varepsilon^t g\, d\mu_{\lambda,\varepsilon} = g(\rho_t)$$

by equicontinuity of $\int P_\varepsilon^t g\, d\mu_{\lambda,\varepsilon}$ as a function of time. The case of deterministic initial data, as stated in Theorem 1, reduces to this average via the continuous dependence of solutions on initial data.

Remarks. There are several ways to generalize Theorem 1. We may add a quadratic nearest neighbor interaction to the energy function $H(S)$ in (1.11):

$$H(S) = \sum_x V(S(x)) + \gamma \sum_{\langle xy \rangle} [S(x) - S(y)]^2 \tag{2.33}$$

The local equilibrium measures $\mu_{\lambda,\varepsilon}$ are no longer product measures in this case. The law of large numbers is still verified, however, if $\gamma \geq 0$ by the convexity of the self-potential V (see also refs. 5 and 17).

The condition that the dimension $d = 1$ can be easily removed (see ref. 5).

Vector-valued models (in the notation of section 1: $n > 1$) can be investigated without any additional difficulty, but we have to assume that the matrix of second derivatives of the nonquadratic part of the energy V is positive definite and close to a constant.

The external field may depend on several neighboring coordinates of the configuration.

We can also add a reaction term $R(S_t^\varepsilon(x))\,dt$ to the right-hand side of Eq. (1.7). Accordingly, the term $\int R(y)\,dq_{\lambda(t,x)}(dy)$ has to be added on the right-hand side of the limiting equation (1.10).

It is possible to introduce spatially inhomogeneous or random conductivities. There is no additional difficulty if the inhomogenity is macroscopic. The problem of microscopic i.i.d. conductivities is not understood.

3. ANHARMONIC SYSTEMS IN NOISE

The configurations of this model are interpreted as couples of elements of Ω_ε, i.e., step functions p^ε and r^ε of step size ε in L_e^2. The evolution law is given in (1.16) by two coupled SDEs. It determines the stochastic processes (p_t^ε) and (r_t^ε) with initial conditions p_0^ε and r_0^ε.

Theorem 2. Suppose that $\phi(p_0^\varepsilon) \to \phi(\Pi_0)$ and $\phi(r_0^\varepsilon) \to \phi(\rho_0)$ as $\varepsilon \downarrow 0$ for each $\phi \in L_e^{2*}$, where Π_0 and $\rho_0 \in H_e^1$; then $\phi(p_t^\varepsilon) \to \phi(\Pi_t)$ and $\phi(r_t^\varepsilon) \to \phi(\rho_t)$ as $\varepsilon \downarrow 0$ in probability for each ϕ, $\phi' \in L_e^{2*}$, where Π_t and ρ_t are determined as weak solutions to (1.20).

This result is a particular case of an extension of Theorem 1 to vector-valued spins; cf. (1.14)–(1.15). Since the additional difficulties are only in the notation, we restrict ourselves here to the proof of Theorem 1.

Remarks. The additional non-Hamiltonian terms of (1.16) are very arbitrary: we could replace $\Delta_\varepsilon p(x)$ and $\Delta_\varepsilon V'(r(x))$ by $\Delta_\varepsilon V_1'(p(x))$ and $\Delta_\varepsilon V_2'(r(x))$ provided that V_1 and V_2 satisfy (2.15). In this case, the coefficients of the limiting equation also depend on V_1 and V_2, but they do not depend on α or β. This instability indicates that stochastic perturbations modify even the average behavior of Hamiltonian dynamics in a substantial way.

The more familiar case of $\beta = 0$ is beyond the scope of our methods. The instability mentioned above seems to be less radical in this situation.

We now give the contents of the remaining sections. In Section 4 the *a priori* bounds are given for the solutions of the backward equation associated to (1.7). Furthermore, the solution of the limiting equation (1.10) is investigated. Both problems rely on the derivation of energy inequalities. The *a priori* bounds are used in Section 5 to establish the relative compactness, in a suitably chosen topology, of families of functions as they appear in the resolvent equation to (2.29). In particular, the L_r^4 estimate will reduce this problem to the case where $b(x, t) = 0$ [see (2.31)] and the method of singular integrals can be used.[5,14] Section 6 discusses the applications of the law of large numbers. Here we find the motivation for the choice of the topology in Section 5. In Section 7 the resolvent method is used to solve the question in the case of random initial data. Section 8 contains the proof of the main theorem. (Theorem 1).

4. THE ENERGY INEQUALITIES

We want to exploit the nice parabolic structure of the microscopic dynamics to develop L_r^2 and L_4^4 estimates. They will give a precise meaning to the statement that the dependence of the evolution on the initial data is smooth.

Let $h \in \Omega_\varepsilon$, and $|r\varepsilon| \leqslant 1$, $z_0 \geqslant 0$. Suppose that a, $b \in \Omega_\varepsilon$, with, for all $x \in R$, $t \geqslant 0$,

$$|a(t, x) - 1| \leqslant \alpha < 1 \tag{4.1}$$

and

$$|b(t, x)| \leqslant \beta \tag{4.2}$$

where α and β are constants.

Lemma 1. Let

$$L_{a,b} v = -z_0 v - \tfrac{1}{2} \nabla_\varepsilon (a \nabla_\varepsilon^* v) + \nabla_\varepsilon (bv) \qquad \text{for} \quad v\colon R \to R \tag{4.3}$$

There exists a constant $C = C(\alpha)$ and a constant $K = K(\alpha, \beta)$ such that

$$\begin{aligned} 2\langle v, L_{a,b} v + \nabla_\varepsilon h \rangle_r &+ 2z_0\, |v|_r^2 + \alpha(1-\alpha)\, |\nabla_\varepsilon^* v|_r^2 \\ &\leqslant (2Cr^2 + K)\, |v|_r^2 + C\, |h|_r^2 \end{aligned} \tag{4.4}$$

Proof. We apply an integration by parts, $\int f \nabla_\varepsilon g = \int (\nabla_\varepsilon^* f)\, g$, and use the identity

$$\nabla_\varepsilon^* (v \theta_r) = (\nabla_\varepsilon^* v)\, \theta_r + (\nabla_\varepsilon^* \theta_r)\, v(\cdot - \varepsilon)$$

to obtain that

$$\begin{aligned} &2\langle v, \nabla_\varepsilon (h - \tfrac{1}{2} a \nabla_\varepsilon^* v + bv) \rangle_r \\ &\quad = 2\langle \nabla_\varepsilon v, (h - \tfrac{1}{2} a \nabla_\varepsilon^* v + bv) \rangle_r \\ &\qquad + 2 \int \varepsilon^{-1} [\theta_r(x-\varepsilon) - \theta_r(x)]\, v(x-\varepsilon)(h - \tfrac{1}{2} a \nabla_\varepsilon^* v + bv)\, dx \end{aligned} \tag{4.5}$$

We now proceed as in ref. 5 and use the basic properties of the weight function θ_r:

$$|\theta_r(x-\varepsilon)-\theta_r(x)| \leqslant |r\varepsilon|\, e^{|r\varepsilon|}\theta_{r/2}(x-\varepsilon)\,\theta_{r/2}(x) \tag{4.6}$$

$$[\theta_{r/2}(x)]^2=\theta_r(x) \tag{4.7}$$

the Schwarz inequality, and the fact that $|r\varepsilon| \leqslant 1$.

Lemma 2. Suppose that $u_s(x)$, $0 \leqslant s \leqslant T \leqslant \infty$, satisfies

$$-\partial_s u_s = -z_0 u_s + h_s + \tfrac{1}{2} a \Delta_\varepsilon u_s + b\nabla_\varepsilon u_s \tag{4.8}$$

with $2z_0 > 2Cr^2+K$ (the constants as in Lemma 1). Then there is a $z>0$ such that

$$|\nabla_\varepsilon u_0|_r^2 + \alpha(1-\alpha)\int_0^T e^{-zs}\,|\Delta_\varepsilon u_s|_r^2\,ds \leqslant e^{-zT}\,|\nabla_\varepsilon u_T|_r^2 + C\int_0^T |h_s|_r^2\,e^{-zs}\,ds \tag{4.9}$$

and

$$|u_0|_r^2 \leqslant e^{-zT}(|u_T|_r^2+|\nabla_\varepsilon u_T|_r^2)+2C\int_0^T |h_s|_r^2\,e^{-zs}\,ds \tag{4.10}$$

Proof. Let $v_s=\nabla_\varepsilon u_s$. Notice that $-\partial_s\,|v_s|^2$ is the lhs of (4.4). This implies the first inequality (4.9) via a straightforward manipulation.

On the other hand, we can multiply the backward equation (4.8) with $2u_s\theta_r$ and get

$$-\partial_s\,|u_s|_r^2+2z_0\,|u_s|_r^2=2\langle u_s,h_s\rangle_r+\langle u_s, a\Delta_\varepsilon u_s\rangle_r+2\langle u_s, b\nabla_\varepsilon u_s\rangle_r \tag{4.11}$$

whence (4.10) follows by (4.9) and the Schwarz inequality.

In the following lemma we are going to derive a similar L_r^4 estimate. We use the notation

$$|f|_{r,p}^p=\int \theta_r(x)\,|f(x)|^p\,dx \tag{4.12}$$

Lemma 3. Suppose that u_s, $0\leqslant s<\infty$, satisfies the backward equation (4.8) with $z_0>Z(\alpha,b,r)$ [defined in (4.26)]. Then there is a $z>0$ and a constant $C'=C'(\alpha)$ such that

$$e^{-zs}\,|\nabla_\varepsilon u_s|_{r,4}^4 \leqslant C'\int_s^\infty e^{-zt}\,|h_t|_{r,4}^4\,dt \tag{4.13}$$

Proof. We start again from the backward equation (4.8), with $v_s = \nabla_\varepsilon u_s$:

$$-\partial_s |v_s|_r^4 = -4z_0 |v_s|_r^4 + 4\langle v_s^3, \nabla_\varepsilon(h_s - \tfrac{1}{2}a\nabla_\varepsilon^* v_s + bv_s)\rangle_r \tag{4.14}$$

In order to integrate by parts the last term in this equation, we note that

$$\nabla_\varepsilon^*(v_s^3\theta_r) = (\nabla_\varepsilon^* v_s)\, F_{s,\varepsilon}\theta_r + \nabla_\varepsilon^*\theta_r v_s^3(\cdot - \varepsilon) \tag{4.15}$$

where

$$F_{s,\varepsilon}(x) = v_s^2(x-\varepsilon) + v_s(x)\, v_s(x-\varepsilon) + v_s^2(x) \geqslant 0 \tag{4.16}$$

After the integration by parts we use the following bounds:

(i)

$$\int \nabla_\varepsilon^*\theta_r(x)\, v_s^3(x-\varepsilon)\, h_s(x)\, dx$$

$$\leqslant |r|\, e^{|r\varepsilon|} e^{|r\varepsilon|/4} \int v_s^3(x-\varepsilon)\, h_s(x)\, \theta_{r/2}(x-\varepsilon)\, \theta_{r/4}(x)\, \theta_{r/4}(x-\varepsilon) \tag{4.17}$$

$$\leqslant 4\,|r|\, |v_s|_{r,4}^3\, |h_s|_{r,4} \quad \text{(by the Hölder inequality)} \tag{4.18}$$

$$\leqslant 3\,|r|^{4/3}\, |v_s|_{r,4}^4 + |h_s|_{r,4}^4 \quad \text{(via the concavity of the log)} \tag{4.19}$$

(ii) [see (i)]

$$\left| \int \nabla_\varepsilon^*\theta_r v_s^3(x-\varepsilon)\, bv_s(x) \right| \leqslant \beta(3 + r^4)\, |v_s|_{r,4}^4 \tag{4.20}$$

(iii)

$$\int \nabla_\varepsilon^*\theta_r v_s^3(x-\varepsilon) \left[-\frac{a}{2} \nabla_\varepsilon^* v_s(x) \right]$$

$$\leqslant \frac{1-\alpha}{2} e^{|r\varepsilon|} \int \theta_{r/2}(x-\varepsilon)\, \theta_{r/2}(x)\, v_s^2(x-\varepsilon) \left[\frac{27r^2}{2} v_s^2(x-\varepsilon) + \frac{|\nabla_\varepsilon^* v_s|^2}{54} \right]$$

$$\leqslant 243(1-\alpha)\, r^2\, |v_s|_{r,4}^4 + \tfrac{1}{12}(1-\alpha) \int \theta_r(x)\, |\nabla_\varepsilon v_s(x)|^2\, v_s^2(x-\varepsilon)$$

$$\leqslant 243(1-\alpha)\, r^2\, |v_s|_{r,4}^4 + \tfrac{1}{6}(1-\alpha) \int \theta_r(x)\, |\nabla_\varepsilon v_s(x)|^2\, F_{\varepsilon,s}(x) \tag{4.21}$$

(iv)

$$\left|\int \theta_r(\nabla_\varepsilon^* v_s)\, F_{\varepsilon,s} h_s\right|$$

$$\leqslant \frac{3}{2(1-\alpha)} \int \theta_r F_{\varepsilon,s} h_s^2 + \frac{1-\alpha}{6} \int \theta_r F_{\varepsilon,s}\, |\nabla_\varepsilon^* v_s|^2$$

$$\leqslant \frac{3}{4}\, |v_s|_{r,4}^4 + \frac{15}{4(1-\alpha)^2}\, |h_s|_{r,4}^4 + \frac{1-\alpha}{6} \int \theta_r F_{\varepsilon,s}\, |\nabla_\varepsilon^* v_s|^2 \tag{4.22}$$

(v)

$$\int \theta_r(\nabla_\varepsilon^* v_s)\, F_{\varepsilon,s}\left(-\frac{a}{2}\nabla_\varepsilon^* v_s\right) \leqslant \frac{\alpha-1}{2} \int \theta_r\, |\nabla_\varepsilon^* v_s|^2\, F_{\varepsilon,s} \tag{4.23}$$

(vi)

$$\left|\int \theta_r(\nabla_\varepsilon v_s)\, F_{\varepsilon,s} b v_s\right|$$

$$\leqslant \frac{3\beta^2}{2(1-\alpha)} \int \theta_r v_s^2 F_{\varepsilon,s} + \frac{1-\alpha}{6} \int \theta_r F_{\varepsilon,s}\, |\nabla_\varepsilon v_s|^2$$

$$\leqslant \frac{21\beta^2}{2(1-\alpha)} \int \theta_r v_s^2 F_{\varepsilon,s} + \frac{1-\alpha}{6} \int \theta_r F_{\varepsilon,s}\, |\nabla_\varepsilon v_s|^2 \tag{4.24}$$

Combining (i)–(vi) with (4.16), we obtain

$$-\partial_s\, |v_s|_{r,4}^4 \leqslant (-4z_0 + 4Z)\, |v_s|_{r,4}^4 + C'\, |h_s|_{r,4}^4 \tag{4.25}$$

where

$$Z = Z(\alpha, \beta, r) = \frac{3}{4} + (3 + r^4)\,\beta + \frac{21\beta^2}{2(1-\alpha)} + 3\,|r|^{4/3} + \frac{243}{4}\,(1-\alpha)\, r^2 \tag{4.26}$$

and

$$C' = C'(\alpha) = \frac{15}{(1-\alpha)^2} + 4$$

This completes the proof.

The *a priori* bounds above will be used to conclude the weak continuity of $f_{z,\varepsilon}$ and $\nabla_\varepsilon Df_{z,\varepsilon}$ via the following result.

Lemma (Lemma 6 in ref. 5). Let B denote an arbitrary ball in L_e^2. For each $\beta > 0$, $r > 0$, and $K < \infty$, there exists a weak neighborhood of 0 in

L_e^2, $U_\alpha(\phi_1,..., \phi_n)$, such that $|\phi(\delta)| < \beta$ whenever $\phi \in \Omega_\varepsilon$, $|\phi|_{-r} + |\nabla_\varepsilon \phi|_{-r} \leqslant K$ and $\delta \in B \cap U_\alpha(\phi_1,..., \phi_n)$.

Proof. See Lemma 6 in ref. 5.

The energy inequalities of Lemma 2 are the discretized versions of the more familiar bounds for the limiting equation (1.10) (see Lemma 4 below). We use the notation of (2.20)–(2.22).

Lemma 4. Given $\rho_0 \in H_e^1$, there exists a continuous trajectory $\rho(t, \cdot)$ in L_e^2, $t \geqslant 0$, such that

$$\rho(0, \cdot) = \rho_0$$

and

$$\int \phi(x)\, \rho(t, x)\, dx$$
$$= \int \phi(x)\, \rho_0(x)\, dx + \int_0^t \int [\tfrac{1}{2} Q(\rho(s, x))\, \phi''(x) + B(\rho(s, x))\, \phi'(x)]\, dx\, ds \tag{4.27}$$

for each twice differentiable $\phi: R \to R$ with compact support. Moreover, $Q(\rho(t, \cdot) = \lambda_t \in H_e^1$ for all $t \geqslant 0$, and $|\lambda_t'|_r^2$ is integrable on bounded time intervals. There is no other solution with such properties.

Proof. Suppose ρ_t is a classical solution to (1.10). One can construct these via a Galerkin approximation. Then,

$$\partial_t\, |\rho_t|_r^2 = -\int \partial_t(\rho_t \theta_r)\, D(\rho_t)\, \rho_t' + 2 \int \rho_t(x)\, \theta_r(x)\, \partial_x B(\rho_t(x)) \tag{4.28}$$

We can obtain some energy inequalities for (4.28) in much the same way as we did for the discrete case (4.8). By the definitions (2.21), $D(\rho_t) = 1/F''(\lambda_t) \geqslant (1-\alpha)^{1/2}$ and $B'(\rho_t(x)) = A'(\lambda_t)\, D(\rho_t) \leqslant c(1-\alpha)^{-1}$, where c is a constant (since J' is bounded). Here we used Proposition 1 of ref. 5: F' is continuously differentiable and $1 - \alpha \leqslant F''(v) \leqslant (1-\alpha)^{-1/2}$ for all $v \in R$.

Therefore, with $r > 0$,

$$\partial_t\, |\rho_t|_r^2 \leqslant -(1-\alpha)^{1/2}\, |\rho'|_r^2 + (r + 2c)(1-\alpha)^{-1}\, |\rho|_r\, |\rho'|_r \tag{4.29}$$

and

$$|\rho_t|_r^2 + \frac{1-\alpha}{2} \int e^{c(t-s)}\, |\rho_s'|_r^2\, ds \leqslant e^{\bar{c}t}\, |\rho_0|_r^2 \tag{4.30}$$

where $\bar{c} = (r + 2c)^2/2(1-\alpha)^3$. Starting from (2.20), it is easy to verify that a similar energy inequality is satisfied by λ':

$$|\lambda_t'|_r^2 + \frac{1-\alpha}{2} \int e^{c(t-s)} |\lambda_s''|_r^2 \, ds \leqslant e^{ct} |\lambda_0'|_r^2 \tag{4.31}$$

The *existence* of the solution in Lemma 4 now follows from a standard argument. We first take ρ_0 and λ_0' in bounded domains of L_e^2. Conditions (4.30) and (4.31) imply compactness properties of the associated classical solutions ρ_t, i.e., ρ_t remains in a strongly compact set of L_e^2 on finite time intervals (by the F. Riesz criterion) because $|\rho_t|_r$ and $|\lambda_t'|_r$ are bounded. (4.31) gives a uniform bound for the time integral of $|(d/dt)\,\rho_t|_r^2$, whence ρ_t is an equicontinuous family in L_e^2. Therefore, the Arzela–Ascoli theorem implies the existence of a continuous trajectory ρ_t in L_e^2 satisfying (4.27) for $\rho_0, \lambda_0' \in L_e^2$. Moreover, (4.30) and (4.31) imply that $\rho_t \in H_e^1$ for all $t \geqslant 0$ and $\int_0^t |\rho_s'|_r^2 \, ds$ and $\int_0^t |\lambda_s'|_r^2 \, ds$ are bounded $[\lambda = Q(\rho)]$.

To prove the *uniqueness* of such a solution, we introduce the current of ρ. Let

$$\omega_t(x) = \int_0^t \left\{ \frac{1}{2} [\lambda_s'(x) - \bar{\lambda}_s'(x)] + B(\rho_s) - B(\bar{\rho}_s) \right\} ds$$

if ρ_t and $\bar{\rho}_t$ are two solutions with $\rho_0 = \bar{\rho}_0$. ω_t is differentiable in the weak sense: for smooth ϕ, $\int \phi' \omega_t \, dx = -\int \phi \delta_t \, dx$ with $\delta_t = \rho_t - \bar{\rho}_t$. We thus have

$$\begin{aligned} \frac{d}{dt} |\omega_t|_r^2 &= \int \theta_r \omega_t \{ (\lambda_t' - \bar{\lambda}_t') + 2[B(\rho_t) - B(\bar{\rho}_t)] \} \, dx \\ &\leqslant [k - (1-\alpha)^{1/2}] \, |\delta_t|_r^2 + C(r, \alpha, c) \, |\omega_t|_r \, |\delta_t|_r \\ &\leqslant K \, |\omega_t|_r^2 \end{aligned} \tag{4.32}$$

where k can be chosen small enough so that $k - (1-\alpha)^{1/2} < 0$ and $K < \infty$ depends on k, α, r, and c (=the constant bounding J'). The conclusion is therefore that $\omega_t = 0 = \delta_t$ a.s., hence uniqueness.

5. THE SMOOTH DEPENDENCE ON THE INITIAL DATA

In the present section we study the behavior of some of the key players as they appear in the resolvent approach (to be presented in the next section).

Let $f_{z,\varepsilon}(\sigma) = \int e^{-zt} P_\varepsilon^t \, g(\sigma) \, dt$, for some $z > 0$ and $\sigma \in L_e^2$ equipped with the weak topology. We take g to be of the form

$$g(\sigma) = \psi(\phi_1(\sigma), \ldots, \phi_k(\sigma)),$$
$$\text{with } \psi \text{ smooth}, \quad \phi_1, \phi_i' \in L_e^{2*} \tag{5.1}$$

$\phi(\sigma) = \int \phi(x)\,\sigma(x)\,dx$ and we write $P_\varepsilon^t g(\sigma) = E_\sigma[g(S_t^\varepsilon)] = g_t^\varepsilon(\sigma)$ [see (2.17)].

Lemma 5. The family of functions $\{f_{z,\varepsilon}(\sigma), 0 < \varepsilon < 1\}$ is relatively compact in $C([z_1, z_2] \times B)$ with the uniform topology, where $B = B(\{b_r\})$ is a ball in L_e^2, and $0 < z_1 < z_2 < \infty$.

Proof. Consider two initial data σ and $\bar{\sigma} \in L_e^2$. By definition,

$$f_{\varepsilon,z}(\sigma) - f_{\varepsilon,z}(\bar{\sigma}) = \int_0^1 \int \delta(x)\, Df_{\varepsilon,z}(x, \bar{\sigma} + q\delta)\, dx\, dq \tag{5.2}$$

where

$$Df(x, \tilde{\sigma}) = E_{\tilde{\sigma}}[u_0(x)] \tag{5.3}$$

and

$$u_s(x) = \int_s^\infty \int e^{z0(s-t)} p_{a,b}(s, x; t, y)\, Dg(y, S_t^\varepsilon)\, dy\, dt \tag{5.4}$$

where $p_{a,b}$ was defined in (2.32). We apply the bounds (4.9) and (4.10) with $T = \infty$, $h_t(x) = Dg(x, S_t^\varepsilon)$, to obtain the weak equicontinuity of $f_{\varepsilon,z}(\sigma)$ as a function of the initial configuration σ. The uniform boundedness follows from

$$|f_{z,\varepsilon}(\sigma)| \leqslant |\psi|_\infty / z \tag{5.5}$$

and the equicontinuity in the parameter z is immediately obtained from

$$|\partial_z f_{z,\varepsilon}(\sigma)| \leqslant |\psi|_\infty / z^2 \tag{5.6}$$

This completes the proof of Lemma 5.

The next lemma discusses properties of the functions

$$\nabla_\varepsilon Df_{z,\varepsilon} \colon L_e^2 \times (0, \infty) \to L_e^{2*}$$

Lemma 6. The family of functions $\{\nabla_\varepsilon Df_{z,\varepsilon}(\cdot, \sigma), 0 < \varepsilon < 1\}$ is relatively compact in $C([z_1, z_2] \times B)$ equipped with the uniform topology, where $B = B(\{b_r\})$ is a ball in L_e^2, and $0 < z_1 < z_2 < \infty$.

Proof. The uniform boundedness is a consequence from (4.9) by taking $T = \infty$, $h_t(x) = Dg(x, S_t^\varepsilon)$ and

$$u_s(x) = \int_s^\infty \int e^{z0(s-t)} p_{a,b}(s, x; t, y)\, Dg(y, S_t^\varepsilon)\, dy\, dt$$

[see (4.8) above]. For the weak equicontinuity we consider two initial data σ and $\bar{\sigma}$ with the corresponding quantities a and $\bar{a}$, b and $\bar{b}$ [defined in (2.32)], h and $\bar{h}$, u_s and $\bar{u}_s$ [defined in (5.4)]. The equation for the difference $\delta u_s = u_s - \bar{u}_s$ reads

$$\begin{aligned} -\partial_s \delta u_s = {} & -z_0 \delta u_s + h_s - \bar{h}_s + \tfrac{1}{2}(a_s - \bar{a}_s) \Delta_\varepsilon u_s \\ & + (b_s - \bar{b}_s) \nabla_\varepsilon u_s + \tfrac{1}{2} \bar{a}_s \nabla_\varepsilon \delta u_s + \bar{b}_s \nabla_\varepsilon \delta u_s \end{aligned} \tag{5.7}$$

Taking together the second, third, and fourth terms in the rhs of this equation, we can use (4.9) to write ($r < 0$)

$$|\nabla_\varepsilon u_0 - \nabla_\varepsilon u_0|_r^2 \leqslant k(I_1 + I_2 + I_3) \tag{5.8}$$

where

$$\begin{aligned} I_1 &= \int_0^\infty dt\, e^{-zt} \, |Dg(\cdot, S_t^\varepsilon) - Dg(\cdot, \bar{S}_t^\varepsilon)|_r^2 \, dt \\ I_2 &= \int_0^\infty dt\, e^{-zt} \, |(a_t - \bar{a}_t) \Delta_\varepsilon u_t|_r^2 \\ I_3 &= \int_0^\infty dt\, e^{-zt} \, |(b_t - \bar{b}_t) \nabla_\varepsilon u_t|_r^2 \, dt \end{aligned} \tag{5.9}$$

and k is a constant. We will show that each of these quantities is small whenever $\sigma - \bar{\sigma} = \delta$ is "small" in the weak topology [see (2.4) and Lemma 6 in ref. 5.

1. Let

$$L_r(Dg) = \operatorname*{Sup}_{\sigma, \bar{\sigma}} \frac{|Dg(\cdot, \sigma) - Dg(\cdot, \sigma)|_r}{|\sigma - \bar{\sigma}|_{-r}} < \infty$$

I_1 is bounded by

$$\begin{aligned} I_1 &\leqslant L_r(Dg) \int_0^\infty e^{-zt} \, |S_t^\varepsilon - \bar{S}_t^\varepsilon|_r^2 \, dt \\ &= L_r(Dg) \int \delta(x) \int_0^\infty e^{-zt} \int p_{\tilde{a}, \tilde{b}}(0, x; t, y) \, k(t, y) \, dy \, dt \, dx \end{aligned} \tag{5.10}$$

where

$$\begin{aligned} k(t, y) &= \theta_{-r}(x) [S_t^\varepsilon(y) - \bar{S}_t^\varepsilon(y)] \\ \tilde{a}_t(y) &= [V'(S_t^\varepsilon(y)) - V'(\bar{S}_t^\varepsilon(y))] / [S_t^\varepsilon(y) - \bar{S}_t^\varepsilon(y)] \\ \tilde{b}_t(y) &= [J(S_t^\varepsilon(y)) - J(\bar{S}_t^\varepsilon(y))] / [S_t^\varepsilon(y) - \bar{S}_t^\varepsilon(y)] \end{aligned}$$

There is a function v_t such that $-\nabla_\varepsilon^* v_t = (S_t^\varepsilon - \bar{S}_t^\varepsilon)\, e^{-zt}$ (see Lemma 4 in ref. 5). Notice that

$$\partial_t v_t = L_{\tilde{a},\tilde{b}} v_t$$

where the operator $L_{a,b}$ was defined in Lemma 1 (Section 4) with $z_0 = z$. Hence (4.10) combined with the estimate of Lemma 4 in ref. 5 implies that $\int_0^\infty |k(t, \cdot)|_r\, dt$ is bounded. Therefore, (4.9) and (4.10) guarantee that

$$\int_0^\infty e^{-zt} \int p_{\tilde{a},\tilde{b}}(0, x; t, y)\, k(t, y)\, dy\, dt$$

and its gradient are uniformly bounded. This is sufficient (see Lemma 6 in ref. 5) to control I_1 in the inequality (5.10).

2. A more subtle argument is required to prove that I_2 is small. We write $\theta_r(x) = \theta_{2r}(x)\, \theta_{-r}(x)$ and apply the Hölder inequality with exponents $q/(q-2)$ and $q/2(q > 2)$. We get

$$I_2 \leqslant \int_0^\infty e^{-zt} [J_1(t)]^{(q-2)/q} [J_2(t)]^{2/q} \tag{5.11}$$

with

$$J_1 = \int \theta_{r'}(y)\, |a_t(y) - \tilde{a}_t(y)|^{2q/(q-2)}\, dy, \qquad J_2 = |\Delta_\varepsilon u_t|_{r,q}^q$$

and $r' = -rq/(q-2) > 0$. Furthermore,

$$J_1 = \int \delta(x) \int p_{\tilde{a},\tilde{b}}(0, x; t, y)\, h(t, y)\, dy\, dx \tag{5.12}$$

where

$$h(t, y) = \theta_{r'}(y) [V''(S_t^\varepsilon(y)) - V''(\bar{S}_t^\varepsilon(y))]^{2q/(q-2)} / [S_t^\varepsilon(y) - \bar{S}_t^\varepsilon(y)] \tag{5.13}$$

is uniformly bounded, since V''' is bounded by assumption. We can now repeat the argument used in part 1 of the proof (replacing k by h) to find that J_1 is "small" if δ is. So it is sufficient to prove that the factor J_2 is bounded. We define the integral operators (for $z > 0$)

$$P_{a,b} h(s, x) = \int_s^\infty e^{z(s-t)} \int p_{a,b}(s, x; t, h)\, h(t, y)\, dy\, dt \tag{5.14}$$

It is clear that

$$P_{a,b} h(s, x) = P_a h(s, x) + P_a b(s, x)\, \nabla_\varepsilon P_{a,b} h(s, x) \tag{5.15}$$

where $P_a = P_{a,b=0}$. Let $L_r^q(R_+)$ be the space of locally integrable h: $R_+ \to R$ with norm

$$|h|_{r,q}^+ = \left[\int_0^\infty \int |h(t, y)|^q \, dy \, dt\right]^{1/q} \tag{5.16}$$

We know from Lemma 3 that $\nabla_\varepsilon P_{a,b}$ is a bounded map from $L_r^4(R_+)$ into itself, while Lemma 2 gives an $L_r^2(R_+)$ bound. We thus have a similar bound for each $2 \leqslant q \leqslant 4$ by the Riesz–Thorin interpolation theorem (see ref. 15). On the other hand, Lemma 7 in ref. 5 provides an $L_r^q(R_+)$ estimate for the operator $\Delta_\varepsilon P_a$ for some $q > 2$. The identity (5.15) together with the boundedness of b shows that J_2 is bounded.

3. I_3 can be treated in essentially the same way as I_2. It is actually a little simpler because we only need the $L_r^4(R^+)$ estimate (4.15) on $\nabla_\varepsilon P_{a,b}$ and the boundedness of J''.

Lemma 7. The family of functions $\{\mu_{\lambda,\varepsilon}(P_\varepsilon^t g(\sigma))\}$ is relatively compact in $C((0, \infty))$.

Proof. The boundedness is trivial. We show that $\{\mu_{\lambda,\varepsilon}(P_\varepsilon^t g(\sigma))\}$ is equicontinuous in the time parameter. For $\sigma \in \Omega_\varepsilon$,

$$\begin{aligned} P_\varepsilon^t g(\sigma) - P_\varepsilon^s g(\sigma) &= \int_s^t G_\varepsilon P_\varepsilon^\tau g(\sigma) \, d\tau \\ &= -\left[\int_s^t \int \left[\tfrac{1}{2} \nabla_\varepsilon \lambda^\varepsilon(x) - J(\sigma(x))\right] Dg_\tau^\varepsilon(x, \sigma) \, dx \, d\tau\right] \end{aligned} \tag{5.17}$$

and

$$Dg_\tau^\varepsilon(x, \sigma) = E_\sigma\left[\int p_{a,b}(0, x; \tau, y) \, Dg(y, S_\tau^\varepsilon) \, dy\right]$$

We use (4.10) with $T = \tau$ to get the bound

$$\text{for all } \sigma \qquad |Dg_\tau^\varepsilon|_r^2 \leqslant e^{2Cr^2\tau}(|Dg|_r^2 + |\nabla_\varepsilon Dg|_r^2) \tag{5.18}$$

The Schwarz inequality completes the proof of the lemma.

6. THE LAW OF LARGE NUMBERS

The limiting equation (1.10) is deterministic. In general the crucial step where the fluctuations are eliminated is taken by the law of large numbers. This law permits us to investigate the "almost sure" behavior of physical quantities which are random for each fixed value of the rescaling parameter ε. The rescaling as carried out in (1.5) expresses the idea that many particles undergo a great number of processes (collisions) in a macroscopic time. In the limit as $\varepsilon \downarrow 0$ the evolution has become macroscopic and deterministic because we have been summing over such a large number of events.[2] A more precise formulation of this idea is the subject of this section.

The measure $\mu_{\lambda,\varepsilon}$ was introduced in (2.23). It is a product measure and therefore its behavior is easy to investigate as $\varepsilon \downarrow 0$. In fact, it is a simple application of Proposition 1 in ref. 5 to conclude:

Lemma 8.

$$\lim_{\varepsilon \downarrow 0} \int g(\sigma)\, \mu_{\lambda,\varepsilon}(d\sigma) = g(\rho) \tag{6.1}$$

where $\rho(x) = F'(\lambda(x))$ if $g \in C_w(\Omega \cap B_n)$ for every $\varepsilon > 0$ with (B_n) an increasing sequence of balls, i.e., $\mu_{\lambda,\varepsilon}(B_n) \uparrow 1$ as $n \uparrow \infty$ uniformly in ε.

Remark. We know from lemma 5 (via Ascoli's theorem) that we can select a subsequence $\varepsilon_n \downarrow 0$ such that $f_{z,\varepsilon_n}(\sigma) \to f_z(\sigma)$. In the same way (from Lemma 6) we know that $\nabla_\varepsilon D f_{z,\varepsilon_n}$ converges to the derivative R'_z of a function R_z. The convergence is uniform on compacts $[z_1, z_2] \times B_m$ (as in Lemmas 5 and 6).

Lemma 9. Let $\rho_0(x) = F'(\lambda(x))$.

(a)

$$\lim_{n \uparrow \infty} \int \mu_{\lambda,\varepsilon_n}(d\sigma)\, f_{z,\varepsilon_n}(\sigma) = f_z(\rho_0) \tag{6.2}$$

(b)

$$\lim_{n \uparrow \infty} \int \mu_{\lambda,\varepsilon_n}(d\sigma) \int \left[\tfrac{1}{2}\nabla_\varepsilon \lambda^\varepsilon(x) - J(\sigma(x))\right] \nabla_\varepsilon D f_{z,\varepsilon_n}(x, \sigma)\, dx$$

$$= \int \left[\tfrac{1}{2}\lambda'(x) + A(\lambda(x))\right] R'_z(x, \rho_0)\, dx \tag{6.3}$$

(c)

$$\lim_{\varepsilon \downarrow 0} \int \mu_{\lambda,\varepsilon}(d\sigma)\, P^t_\varepsilon g(\sigma) - P^t_\varepsilon g(\rho^\varepsilon_0) = 0 \tag{6.4}$$

Proof. (a)

$$\left| \int \mu_{\lambda,\varepsilon_n}(d\sigma) f_{z,\varepsilon_n}(\sigma) - f_z(\rho_0) \right|$$
$$\leqslant \left| \int \mu_{\lambda,\varepsilon_n}(d\sigma)[f_{z,\varepsilon_n}(\sigma) - f_z(\sigma)] \right|$$
$$+ \left| \int \mu_{\lambda,\varepsilon_n}(d\sigma) f_z(\sigma) - f_z(\rho_0) \right| \tag{6.5}$$

The first term is bounded, for every $m \in N$, by

$$\operatorname*{Sup}_{\sigma \in B_m} |f_{z,\varepsilon_n}(\sigma) - f_z(\sigma)| + 2\,|\psi|_\infty\, \mu_{\lambda,\varepsilon_n}(B_m^c)/z \tag{6.6}$$

As a result of Lemma 8 and the remark following this lemma every term vanishes in the limit $n \uparrow \infty$.

(b) $\nabla_\varepsilon \lambda^\varepsilon$ converges strongly to λ'; on the other hand,

$$\int J(\sigma(x))\, \nabla_\varepsilon\, Df_{z,\varepsilon_n}(x, \sigma)\, dx$$
$$= \int J(\sigma(x))[\nabla_\varepsilon\, Df_{z,\varepsilon_n}(x, \sigma) - R'_z(x, \rho_0)]\, dx$$
$$+ \int J(\sigma(x))\, R'_z(x, \rho_0)\, dx \tag{6.7}$$

For the first term we can apply the Schwarz inequality together with the argument of part (a) of this lemma. The second term converges trivially and gives the desired result (6.4), with $R'_z(\cdot\,, \rho_0) \in L_e^{2*}$.

(c) Given Lemma 8, it is sufficient to prove that $g_t^\varepsilon(\sigma) - g_t^\varepsilon(\bar{\sigma})$ is small whenever $\delta = \sigma - \bar{\sigma}$ is "small" (in the usual "weak" sense). But,

$$g_t^\varepsilon(\sigma) - g_t^\varepsilon(\bar{\sigma}) = \int \delta(x) \int_0^1 Dg_t^\varepsilon(x, \bar{\sigma} + \delta q)\, dx\, dq \tag{6.8}$$

with

$$Dg_t^\varepsilon(x, \bar{\sigma}) = E_{\bar{\sigma}}\left[\int p_{a,b}(0, x; t, y)\, Dg(y, S_t^\varepsilon) \right]$$

We can repeat the argument already applied in the proof of Lemma 7: it shows that $|Dg_t^\varepsilon(x, \sigma)|_r^2$ and $|\nabla_\varepsilon\, Dg_t^\varepsilon(x, \sigma)|_r^2$ are bounded. This is sufficient to complete the proof (see Lemma 6 in ref. 5).

7. THE RESOLVENT APPROACH

We are now in a position to apply the resolvent method. The power of this method in the context of the derivation of the hydrodynamic limit was very well established in refs. 5, 12, 18, and 19. It extends to positive times the expected behavior for the macroscopic quantities which can be guessed from the form of the generator [see (2.26)]. An alternative approach (at least for finite volumes, or on the torus) was recently suggested in ref. 20.

From the definition of the resolvent $f_{z,\varepsilon}(\sigma)$ as it was introduced in (5.1), we obtain the (resolvent) equation

$$g(\sigma) = zf_{z,\varepsilon}(\sigma) - G_\varepsilon f_{z,\varepsilon}(\sigma) \tag{7.1}$$

We can integrate this equation with the measure $\mu_{\lambda,\varepsilon}(d\sigma)$. Lemma 9 asserts that we can pass to a limiting resolvent equation along some subsequence ε_n:

$$g(F'(\lambda)) = zf_z(F'(\lambda)) - Gf_z(F'(\lambda)) \tag{7.2}$$

for each $\lambda \in H_e^1$ and $z > z_0$, where z_0 depends on g only. The uniqueness of the resolvent equation and Lemma 4 imply that

$$f_z(\rho_0) = \int_0^\infty e^{-zt} g(F'(\lambda_t))\, dt \tag{7.3}$$

where λ_t was defined in (2.20). The uniqueness of the Laplace transform together with Ascoli's theorem [which can be used via the result of Lemma 9(c)] are sufficient to conclude:

Lemma 10. For all $\lambda \in H_e^1$,

$$\lim_{\varepsilon \downarrow 0} \mu_{\lambda,\varepsilon}(P_\varepsilon^t g) = g(F'(\lambda_t)) \tag{7.4}$$

where λ_t is defined in (2.20).

8. PROOF OF THEOREM 1

The previous section solved the problem in the case of random initial data. We now have to pass to a deterministic initial condition.

By assumption, σ^ε converges weakly in L_e^2 to $\rho_0 \in H_e^1$. Given $\rho_0 \in H_e^1$, we find $\lambda \in H_e^1$ via the relation

$$\lambda(x) = Q(\rho_0(x)) \tag{8.1}$$

This allows us to construct the measures $\mu_{\lambda,\varepsilon}(d\sigma)$ on the initial configurations $\sigma \in \Omega_\varepsilon$ [as was shown in (2.23)]. Let ρ_t be the solution to (4.27). With the function g as in (5.1),

$$\begin{aligned}|g_t^\varepsilon(\sigma^\varepsilon) - g(\rho_t)| &\leqslant |g_t^\varepsilon(\sigma^\varepsilon) - g_t^\varepsilon(\rho_0^\varepsilon)| + \left| g_t^\varepsilon(\rho_0^\varepsilon) - \int \mu_{\lambda,\varepsilon}(d\sigma)\, g_t^\varepsilon(\sigma)\right| \\ &\quad + \left|\int \mu_{\lambda,\varepsilon}(d\sigma)\, g_t^\varepsilon(\sigma) - g(\rho_t)\right| \end{aligned} \tag{8.2}$$

Since $\phi(\sigma^\varepsilon) \to \phi(\rho_0)$ for each $\phi \in L_e^{2*}$, the first term goes to zero by the argument given in the proof of part (c) of Lemma 9. The second term approaches zero as $\varepsilon \downarrow 0$ as a direct consequence of Lemma 9(c). Finally, the third term was treated in Lemma 10. The conclusion of Theorem 1 now follows from a Chebyshev inequality.

ACKNOWLEDGMENTS

We are greatly indebted to Joel L. Lebowitz for suggesting the problem, and for stimulating discussions and useful remarks. This work was partially supported by NSF grant DMR 86-12369. J.F. was supported in part by the Hungarian Foundation for Scientific Research No. 819/1.

REFERENCES

1. A. De Masi, N. Ianiro, S. Pellegrinotii, and E. Presutti, in *Non-Equilibrium Phenomena II, From Stochastics to Hydrodynamics*, J. L. Lebowitz and E. W. Montroll, eds. (North-Holland, Amsterdam, 1984).
2. J. L. Lebowitz, E. Presutti, and H. Spohn, Microscopic models of hydrodynamical behavior, Preprint.
3. C. Boldrighini, R. L. Dobrushin, and Yu. M. Suhov, One-dimensional hard rod caricature of hydrodynamics, *J. Stat. Phys.* **31**:577 1983).
4. R. L. Dobrushin, S. Pellegrinotti, Yu. M. Suhov, and L. Triolo, *J. Stat. Phys.* **43**:571 (1986).
5. J. Fritz, On the hydrodynamic limit of a Ginzburg–Landau lattice model. The law of large numbers in arbitrary dimensions, preprint.
6. J. W. Cahn and J. E. Hilliard, *J. Chem. Phys.* **28**:258 (1958).
7. B. I. Halperin and P. C. Hohenberg, *Rev. Mod. Phys.* **49**:435 (1977).
8. A. De Masi, E. Presutti, and E. Scacciatelli, The weakly asymmetric simple exclusion process, Preprint.
9. J. Gärtner, Convergence towards Burger's equation and propagation of chaos for weakly asymmetric exclusion processes, Preprint.
10. J. Krug, J. L. Lebowitz, H. Spohn, and M. Q. Zhang, *J. Stat. Phys.* **44**:535 (1986).
11. H. van Beijeren, R. Kutner, and H. Spohn, Excess noise for driven diffusive systems. *Phys. Rev. Lett.* **54**:2026 (1985).

12. J. Fritz, in *IMA Volumes in Mathematics* (1986), Vol. 9, p. 75.
13. A. De Masi, P. A. Ferrari, and J. L. Lebowitz, *J. Stat. Phys.* **44**:589 (1986).
14. D. Stroock and S. R. S. Varadhan, *Multidimensional Diffusion Processes* (Springer-Verlag, New York, 1979).
15. E. M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces* (Princeton University Press, 1971).
16. K. Yoshida, *Functional Analysis* (Springer-Verlag, 1980).
17. H.-R. *Commun. Math. Phys.* **84**:207 (1982).
18. J. Fritz, *J. Stat. Phys.* **47**:551 (1987).
19. T. Funaki, Derivation of the hydrodynamical equation for a 1-dimensional Ginzburg–Landau model, Preprint.
20. M. Z. Guo, G. Papanicolau, and S. R. S. Varadhan, Nonlinear diffusion limit for a system with nearest neighbor interactions, Preprint.